Monge Problem on infinite dimensional Hilbert space endowed with suitable Gaussian measure

Vincent NOLOT

Institut de Mathématiques de Bourgogne, Université de Bourgogne, 21078 Dijon, France.
vincent.nolot@u-bourgogne.fr

Abstract

In this paper we solve the Monge problem on infinite dimensional Hilbert space endowed with a suitable Gaussian measure.

1 Introduction

Our framework is an infinite dimensional Hilbert space \((H, \langle \cdot, \cdot \rangle)\) endowed with its Borelian \(\sigma\)-algebra. For \(\rho_0\) and \(\rho_1\) two Borel probability measures on \(H\), the Monge Problem consists of finding a Borel map \(T : H \to H\) satisfying the constraint \(T \# \rho_0(B) := \rho_0(T^{-1}(B)) = \rho_1(B)\) (for any Borel subset \(B\) of \(H\)) and minimizing the quantity

\[
\int_H c(x, T(x)) d\rho_0(x),
\]

where \(c : H \times H \to [0, \infty)\) is called cost function.

Theorem 1.1 Assume that \(\rho_0\) and \(\rho_1\) have finite relative entropy with respect to \(\gamma\) where \(\gamma\) satisfies conditions of Theorem 1.2. Then the problem

\[
\inf_{T \# \rho_0 = \rho_1} \int_H |x - T(x)| d\rho_0(x)
\]

has at least one solution \(T : H \to H\).

Monge Problem has been solved in infinite dimensional Hilbert spaces, when the cost is \(c(x, y) = |x - y|^p\) and \(p > 1\) (see e.g. [?]). The case when \(p\) is equal to 1 is quite more tricky. This is the object of our paper.

We are inspired from Champion and De Pascale in [?]. The strategy for infinite dimensional case lies on the same powerful tool as in finite dimensional case: an essential ingredient is the differentiation theorem for the measure of reference. Unfortunately there is some measure on Hilbert spaces for which this theorem is false. Nevertheless Tiser has proved in [?] that for a suitable Gaussian measure on some Hilbert space, the differentiation theorem holds, namely:

Theorem 1.2 Let \(H\) be a separable Hilbert space and let \(\gamma\) be a Gaussian measure with the following representation of its covariance operator :

\[
R(x) = \sum_i c_i(x, e_i)e_i,
\]
where \((c_i)_i\) is an orthonormal system of \(H\). Suppose that for \(\alpha > 5/2\) given we have \(c_{i+1} \leq c_i/\alpha\) for all \(i\). Then
\[
\lim_{r \to 0} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |f - f(x)| d\gamma = 0 \quad \text{for } \gamma - \text{a.a. } x \in H
\]
for any \(f \in L^p(H,\gamma)\) and \(p > 1\).

The set of \(x \in H\) such that Theorem 1.2 holds, is called the set of Lebesgue points of \(f\) and will be denoted by \(\text{Leb}(f)\). Thus \(\gamma(\text{Leb}(f)) = 1\). In the case of \(f = \mathbb{I}_A\), we will call \(x\) a Lebesgue point of \(A\).

**Remark 1.3** In fact the Theorem 1.2 is required only to get the Proposition 4.4. All other results in this section are available without Lebesgue points.

From now, \(\gamma\) is the Gaussian measure defined on \(H\) satisfying conditions of the previous Theorem 1.2. So that the differentiation theorem holds over \((H,\gamma)\).

The classical way to find a solution of (1) is to introduce the following Monge-Kantorovich problem:
\[
\min_{\Pi \in C(\rho_0,\rho_1)} \int_{H \times H} |x - y| d\Pi(x,y), \quad (2)
\]
where \(C(\rho_0,\rho_1)\) is the set of coupling between \(\rho_0\) and \(\rho_1\). The nonempty set of solutions (optimal couplings) of (2) will be denoted by \(C_1(\rho_0,\rho_1)\). Among these coupling, we shall show there is at least one which is carried by a graph of some map \(T\) and therefore this map will be a solution of (1).

Because the cost induced by the euclidian norm is not strictly convex, the set \(O_1(\rho_0,\rho_1)\) does not contain enough information to construct some map \(T\). Thus we need introduce another problem, called second variational problem, with a new cost to minimize over the set of optimal couplings of (2):
\[
\min_{\Pi \in C_1(\rho_0,\rho_1)} \int_{H \times H} \alpha(x - y) d\Pi(x,y), \quad (3)
\]
with
\[
\alpha(x - y) := \sqrt{1 + |x - y|^2}.
\]
This cost \(\alpha\) is strictly convex and smooth. It turns out that it shall bring more information, namely in some sense the directions that should take the optimal plan in order to be concentrated on a graph of some map.

We denote by \(O_2(\rho_0,\rho_1)\) the subset of \(O_1(\rho_0,\rho_1)\) containing optimal couplings which minimize (3). It is easy to see that \(\alpha(x - y) \leq 1 + |x - y|\) so that if (2) is finite for some coupling then (3) is also finite, and the set \(O_2(\rho_0,\rho_1)\) is a nonempty (by weak compactness) and a convex subset of \(C(\rho_0,\rho_1)\).

For our purpose, we need to consider finite dimensional approximations. Since the eigenvalues of the covariance matrix of \(\gamma\) are \((c_i)_i\) we can identify \((H,\gamma)\) with \((l^2(c),\mu)\) where \(\mu\) is the product of standard Gaussian measure on \(\mathbb{R}\) and
\[
l^2(c) := \{x \in \mathbb{R}^N, \sum c_i x_i^2 < \infty\}.
\]
This latter space is separable, therefore we consider a sequence of maps \((\pi_n)_n\) such that each \(\pi_n\) projects \(l^2(c)\) onto a \(n\)-dimensional euclidian space, and \(\lim_n \pi_n = \mathbb{I}\) the identity map on \(l^2(c)\). In the sequel we make the abuse of notation \((H,\gamma) = (l^2(c),\mu)\).
We denote by $A_n$ the $\sigma$-algebra generated by $\pi_n$, and if (for $i = 0, 1$) $f_i$ is the density of $\rho_i$ w.r.t. $\mu$, we put $\hat{\rho}_i^n := E[f_i|A_n|\gamma]$. For $x \in H$ and $n \in \mathbb{N}$, we denote by $x_n := \pi_n(x)$.

We say that a coupling $\Pi \in C(\rho_0, \rho_1)$ satisfies the convexity property if the relative entropy is 1-convex along geodesics $\rho_t := ((1-t)P_1 + tP_2)\#\Pi$, namely

$$Ent_\gamma(\rho_t) \leq (1-t)Ent_\gamma(\rho_0) + tEnt_\gamma(\rho_1) - W^2(\rho_0, \rho_1),$$

holds for any $t \in (0, 1)$.

Finally we are interested in the following set:

$$\overline{O}_2(\rho_0, \rho_1) := \{\Pi \in O_2(\rho_0, \rho_1), \Pi \text{ enjoys the convexity property}\}.$$

The fact that $\overline{O}_2(\rho_0, \rho_1)$ is non empty is the purpose of Theorem 2.5. It will play a key role in our approach because any coupling of $\overline{O}_2(\rho_0, \rho_1)$ will bring us enough information to show that it is concentrated on a graph of some measurable map.

Let us present how this paper is organized. In section 2 we establish the convexity of relative entropy (w.r.t. $\mu$) in $(P_1(H), W_1)$. In particular we obtain that if $\Pi \in \overline{O}_2(\rho_0, \rho_1)$ then $\rho_t := ((1-t)P_1 + tP_2)\#\Pi$ (here $P_t$ designs the projection onto the $i$-th component) is absolutely continuous w.r.t. $\mu$ for any $t \in (0, 1)$. This point will be necessary through the next sections. In section 3 we present different features of the support of element belonging to $\overline{O}_2(\rho_0, \rho_1)$. Proposition 4.4, which relies on Lemma 4.1 is paramount for the method used in the proof of Theorem 1.1. Finally the last section is devoted to prove Theorem 1.1. The end contains many comments about the proof and about open problems.

## 2 Convexity of relative entropy in $(P_1(H), W_1)$

The following Proposition states that the relative entropy with respect to the Lebesgue measure on $\mathbb{R}^n$ is convex along geodesics in $(P_p(\mathbb{R}^n), W_p)$ whatever $p > 1$. It is fundamental to get all other results of convexity of relative entropy (when the reference measure is absolutely continuous with respect to the Lebesgue measure).

**Proposition 2.1** Let $c$ be a strictly convex and differentiable norm on $\mathbb{R}^n \setminus \{0\}$. If $p > 1$ then for any $\rho_0, \rho_1 \in D(Ent_c)$ and $\Pi$ optimal (for $c$) coupling between $\rho_0$ and $\rho_1$, $\rho_t := (T_t)\#\Pi$ satisfies

$$Ent_c(\rho_t) \leq (1-t)Ent_c(\rho_0) + tEnt_c(\rho_1), \quad \forall t \in [0, 1].$$

**Proof.** See for example [2] (Chapter 5.). \hfill $\square$

In order to extend our result in infinite dimensional spaces, we work with Gaussian measures as reference measures. Let $\gamma_n$ be the standard Gaussian measure on $\mathbb{R}^n$. We consider $\rho_0$ and $\rho_1$ two probability measures on $\mathbb{R}^n$ belonging to $D(Ent_\gamma)$. For the Euclidian norm $|.|$ on $\mathbb{R}^n$, we introduce quantity inspired from the so called Wasserstein distance:

$$W_c(\rho_0, \rho_1) := \inf_{\Pi \in C(\rho_0, \rho_1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y| + \epsilon \alpha(x - y) d\Pi(x, y),$$

where

$$\alpha(x - y) := (1 + |x - y|)^{1/2}.$$ 

Here $\alpha$ is strictly convex and differentiable function on $\mathbb{R}^n$. We have the relation:

$$c_\epsilon(x - y) := |x - y| + \epsilon \alpha(x - y) \leq \epsilon + (1 + \epsilon)|x - y|.$$
Beside $c_ε$ is not a distance, neither is $W_ε$.

We recall that the 1—Wasserstein distance in this situation is defined as

$$W_1(ρ_0, ρ_1) := \inf_{Π ∈ C(ρ_0, ρ_1)} \int_{R^n × R^n} |x - y|Π(x, y).$$

Because of $\lim inf_{ε → 0} W_ε(ρ_0, ρ_1) ≥ W_1(ρ_0, ρ_1)$ we always fix $ε$ small enough in such a way that,

$$W_ε(ρ_0, ρ_1) - ε ≥ W_1(ρ_0, ρ_1) - ε > 0.$$

**Proposition 2.2** If $Π$ is optimal for the cost $c_ε$ then for any $t ∈ (0, 1)$ and:

$$Ent_γ(ρ_i) ≤ (1 - t)Ent_γ(ρ_0) + tEnt_γ(ρ_1) - \frac{t(1 - t)}{2(1 + ε)^2} (W_ε(ρ_0, ρ_1) - ε)^2. \tag{4}$$

In particular if $ρ_0, ρ_1 ∈ D(Ent_γ)$ then also $ρ_t ∈ D(Ent_γ)$ for any $t ∈ (0, 1)$.

**Proof.** We can assume that $ρ_0, ρ_1 ∈ D(Ent_γ)$, otherwise the inequality is obvious. Therefore since $ρ_0$ and $ρ_1$ be two probability measures absolutely continuous with respect to $γ_n$, they are also absolutely continuous with respect to the Lebesgue measure $L$. For $i = 0, 1$ let $dρ_0 = f_0dL$ and $dρ_1 = f_1dL$, then the density of probability of $ρ_i$ with respect to $γ_n$ is $\frac{dρ_i}{γ_n} = f_i(2π)^{\frac{n}{2}} e^{-\frac{|x|^2}{2}}$. Write:

$$Ent_γ(ρ_i) = \int_{R^n} f_i(x)(2π)^{\frac{n}{2}} e^{-\frac{|x|^2}{2}} \log \left( f_i(x)(2π)^{\frac{n}{2}} e^{-\frac{|x|^2}{2}} \right) dγ_n(x)$$

$$= \int_{R^n} f_i(x)(2π)^{\frac{n}{2}} e^{-\frac{|x|^2}{2}} \log(f_i(x))dγ_n(x) + \int_{R^n} f_i(x)(2π)^{\frac{n}{2}} e^{-\frac{|x|^2}{2}} \log((2π)^{\frac{n}{2}})dγ_n(x)$$

$$+ \int_{R^n} f_i(x)(2π)^{\frac{n}{2}} e^{-\frac{|x|^2}{2}} \frac{|x|^2}{2}dγ_n(x)$$

$$= Ent_L(ρ_i) + V(ρ_i) + \frac{d}{2}\log(2π),$$

where $V(ρ_i) := \frac{1}{2} \int |x|^2 dρ_i(x)$. By 1—convexity of the euclidian norm, it is easy to see that:

$$V(ρ_i) ≤ (1 - t)V(ρ_0) + tV(ρ_1) - \frac{t(1 - t)}{2} \int_{R^n} |x - y|^2 dΠ(x, y).$$

By Cauchy-Schwarz inequality, we get

$$V(ρ_i) ≤ (1 - t)V(ρ_0) + tV(ρ_1) - \frac{t(1 - t)}{2(1 + ε)^2} (W_ε, ||)(ρ_0, ρ_1) - ε)^2. \tag{5}$$

then combining the Proposition 2.1 with (5), we get the result taking the sum. □

Now we focus on our separable infinite dimensional Hilbert space ($H, γ$).

In order to apply these results above, we are interested in the following problem:

$$\min_{Π ∈ C(ρ_0, ρ_1)} \int_{H × H} |x - y|dΠ(x, y) + ε \int_{H × H} α(x - y)dΠ(x, y), \tag{P}$$

where $α$ is defined as above,

$$α(x - y) := (1 + |x - y|^{1/2}).$$

Here $|.|$ stands for the Hilbert norm on $H$.

The following result extends the Proposition 2.2 to the infinite dimensional Hilbert space.
Proposition 2.3 Let $\Pi_\varepsilon$ be a solution of $(P_\varepsilon)$, being $w$–limit point of a sequence $(\Pi_{n})_n$ with $\Pi_{n} \in C(\rho_{0}^{n},\rho_{1}^{n})$ optimal for $c_\varepsilon$ and satisfying (4). If $\rho_t := (T_t)_\# \Pi$ then for any $t \in (0, 1)$, $\rho_t \in D(\text{Ent}_\gamma)$ and:

$$\text{Ent}_\gamma(\rho_t) \leq (1-t)\text{Ent}_\gamma(\rho_0) + t\text{Ent}_\gamma(\rho_1) - \frac{t(1-t)}{2(1+\varepsilon)}(W_\varepsilon(\rho_0,\rho_1) - \varepsilon)^2. \quad (6)$$

Proof. Fix $n \in \mathbb{N}$. Define $\rho^n_t := (T_t)_\# \Pi_n$ for $t \in (0, 1)$. Because of all measures $\rho^n_t$ can be seen on probability measures over $H$,

$$\text{Ent}_\gamma(\rho^n_t) = \text{Ent}_\gamma(\rho^n_0) \quad \forall t \in [0, 1],$$

and we apply the Proposition 2.2 in the case of the Euclidian norm $||.||$, that is for all $t \in [0, 1]$: 

$$\text{Ent}_\gamma(\rho^n_t) \leq (1-t)\text{Ent}_\gamma(\rho^n_0) + t\text{Ent}_\gamma(\rho^n_1) - \frac{t(1-t)}{2(1+\varepsilon)}(W_\varepsilon(\rho^n_0,\rho^n_1) - \varepsilon)^2.$$ 

And $W_{\varepsilon,||}(\rho_0,\rho_1) \leq \liminf_n W_{\varepsilon,||}(\rho^n_0,\rho^n_1)$, therefore if $\delta > 0$ is small enough so that $W_{\varepsilon,||}(\rho_0,\rho_1) - \varepsilon - \delta > 0$, we can find $N \in \mathbb{N}$ such that:

$$W_{\varepsilon,||}(\rho^n_0,\rho^n_1) + \delta \geq W_{\varepsilon,||}(\rho_0,\rho_1) \quad \forall n \geq N.$$ 

Jensen’s inequality implies $\text{Ent}_\gamma(\rho^n_t) \leq \text{Ent}_\gamma(\rho_t)$ for $i = 0, 1$. Then for all $n \geq N$:

$$\text{Ent}_\gamma(\rho^n_t) \leq (1-t)\text{Ent}_\gamma(\rho_0) + t\text{Ent}_\gamma(\rho_1) - \frac{t(1-t)}{2(1+\varepsilon)}(W_\varepsilon(\rho_0,\rho_1) - \varepsilon - \delta)^2.$$ 

Since $(\Pi_n)_n$ converges weakly to $\Pi$, it is the same for $(\rho^n_t)_n$ to $\rho_t$, and the compacity of the set $\{\text{Ent}_\gamma(.) \leq R\}$, and the lower semicontinuity of $\text{Ent}_\gamma(.)$ let us to conclude:

$$\text{Ent}_\gamma(\rho_t) \leq (1-t)\text{Ent}_\gamma(\rho_0) + t\text{Ent}_\gamma(\rho_1) - \frac{t(1-t)}{2(1+\varepsilon)}(W_\varepsilon(\rho_0,\rho_1) - \varepsilon - \delta)^2.$$ 

Letting $\delta \to 0$, the result follows. \qed

For the next Corollary, we deal with the true Wasserstein distance $W_{1,||}$ on $\mathcal{P}(H)$. In this case for $\Pi \in O_1(\rho_0,\rho_1)$ we can talk about (constant speed) geodesics for $\rho_t := (T_t)_\# \Pi$, namely

$$W_1(\rho_t,\rho_s) = \lvert t - s \rvert W_1(\rho_0,\rho_1), \quad \forall t \in [0, 1].$$

Corollary 2.4 Let $\Pi \in C(\rho_0,\rho_1)$ be a $w$–limit point of $(\Pi_{\varepsilon})_\varepsilon$ solutions of $(P_{\varepsilon})$, and such that each $\Pi_{\varepsilon}$ satisfies (6). If $\rho_t := (T_t)_\# \Pi$ then for any $t \in (0, 1)$, $\rho_t \in D(\text{Ent}_\mu)$ and:

$$\text{Ent}_\gamma(\rho_t) \leq (1-t)\text{Ent}_\gamma(\rho_0) + t\text{Ent}_\gamma(\rho_1) - \frac{t(1-t)}{2} W_{1,||}^2(\rho_0,\rho_1). \quad (7)$$

In the literature, this proposition can be reformulated as: relative entropy is geodesically 1–convex in $(\mathcal{P}(H), W_{1,||})$.

Proof. Let $\rho^t_{\varepsilon} := ((1-t)P_1 + tP_2)_\# \Pi_{\varepsilon}$. Thanks to the Proposition 2.3:

$$\text{Ent}_\gamma(\rho^t_{\varepsilon}) \leq (1-t)\text{Ent}_\gamma(\rho_0) + t\text{Ent}_\gamma(\rho_1) - \frac{t(1-t)}{2(1+\varepsilon)^2}(W_{\varepsilon,||}(\rho_0,\rho_1) - \varepsilon)^2.$$ 

Because $W_{\varepsilon,||}$ converges to the Hilbert norm $||.||$ when $\varepsilon$ goes to 0, it turns out that $W_{1,||}(\rho_0,\rho_1) \leq \liminf_{\varepsilon \to 0} W_{\varepsilon,||}(\rho_0,\rho_1)$. Arguing as in the proof above, for all $\delta > 0$ and $\varepsilon$ small enough:

$$\text{Ent}_\gamma(\rho_t) \leq (1-t)\text{Ent}_\mu(\rho_0) + t\text{Ent}_\mu(\rho_1) - \frac{t(1-t)}{2(1+\varepsilon)^2}(W_{1,||}(\rho_0,\rho_1) - \varepsilon - \delta)^2.$$ 

5
Finally we let $\varepsilon$ goes to 0 and then $\delta$ goes to 0. \hfill \Box

A particular case of application of the previous Proposition is the following: if $\Pi \in \overline{O}_2(\rho_0, \rho_1)$ then the interpolation $\rho_t := ((1-t)\Pi_1 + t\Pi_2)\pi \Pi$ is absolutely continuous with respect to $\mu$ for any $t \in (0,1)$. We are now able to pick up some elements in $\overline{O}_2(\rho_0, \rho_1)$. 

**Theorem 2.5** $\overline{O}_2(\rho_0, \rho_1)$ is a non empty set.

**Proof.** For all $n \in \mathbb{N}$ and $\varepsilon > 0$ we consider $\Pi_{n,\varepsilon} \in C(\rho^n_0, \rho^n_1)$ optimal for the cost $c_\varepsilon$. It implies that (4) holds for $\Pi_{n,\varepsilon}$. Now we pass to the Hilbert space and up to a subsequence, $(\Pi_{n,\varepsilon})_n$ converges weakly to some coupling $\Pi_\varepsilon \in C(\rho_0, \rho_1)$ which solution of the problem $(P_\varepsilon)$. Therefore (6) holds for $\Pi_\varepsilon$. Again if $\Pi$ is a limit point of $(\Pi_\varepsilon)_\varepsilon$, then again (7) holds for $\Pi$, namely $\Pi$ satisfies the convexity property. We claim that any cluster point of $(\Pi_\varepsilon)_\varepsilon$ belongs to $\overline{O}_2(\rho_0, \rho_1)$. As a consequence, the set $\overline{O}_2(\rho_0, \rho_1)$ will be non empty.

Let $\Pi$ be a limit point of $(\Pi_\varepsilon)_\varepsilon$.

* $\Pi \in O_1(\rho_0, \rho_1)$. Indeed if $\Pi_0 \in O_1(\rho_0, \rho_1)$, for $\varepsilon > 0$:

$$\int |x-y|d\Pi_\varepsilon \leq \int |x-y|d\Pi_0 + \varepsilon \int \alpha(x-y)d\Pi_0.$$ Letting $\varepsilon \to 0$,

$$\int |x-y|d\Pi \leq \liminf_{\varepsilon \to 0} \int |x-y|d\Pi_\varepsilon \leq \int |x-y|d\Pi_0.$$ * $\Pi \in O_2(\rho_0, \rho_1)$. Indeed if $\Pi_0 \in O_2(\rho_0, \rho_1)$, for $\varepsilon > 0$:

$$\int |x-y|d\Pi_\varepsilon + \varepsilon \int \alpha(x-y)d\Pi_\varepsilon \leq \int |x-y|d\Pi_0 + \varepsilon \int \alpha(x-y)d\Pi_0 \leq \int |x-y|d\Pi_\varepsilon + \varepsilon \int \alpha(x-y)d\Pi_0,$$

the latter inequality is provided by the fact that $\Pi_0$ belongs in particular to $O_1(\rho_0, \rho_1)$. Remove the same terms, dividing by $\varepsilon$ and letting $\varepsilon \to 0$,

$$\int \alpha(x-y)d\Pi \leq \liminf_{\varepsilon \to 0} \int \alpha(x-y)d\Pi_\varepsilon \leq \int \alpha(x-y)d\Pi_0.$$ 

Note also that for $\Pi_1$ and $\Pi_2$ are two coupling in $C(\rho_0, \rho_1)$ enjoying the convexity property, every linear combination $(1-t)\Pi_1 + t\Pi_2$ still enjoys the convexity property. As a consequence $\overline{O}_2(\rho_0, \rho_1)$ is a convex set.

### 3 Recalls on optimal transportation theory

We refer to [?] or [?] for proofs of results of this section. We denote by $\text{Supp}(\Pi)$ the *support of $\Pi$, namely the smallest closed subset of $W \times W$ on which $\Pi$ is concentrated.*

**Definition 3.1** Let $(X, \mu)$ and $(Y, \nu)$ be two Polish probability spaces and $c : X \times Y \to [0,\infty]$ be a measurable cost function. We say that $\Pi \in C(\mu, \nu)$ is $c$-cyclically monotone when for any $N \in \mathbb{N}$ and $(x_1, y_1), \ldots, (x_N, y_N) \in \text{Supp}(\Pi)$, we have:

$$\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}),$$
with \( y_{N+1} = y_1 \).

**Proposition 3.2** Let \((X, \mu)\) and \((Y, \nu)\) be two Polish probability spaces and \( c : X \times Y \to [0, \infty) \) be a lower semi-continuous cost function. Then any optimal coupling of

\[
\min_{\Pi \in C(\mu, \nu)} \int_{X \times Y} c(x, y) d\Pi(x, y)
\]

is \( c \)-cyclically monotone.

**Proposition 3.3** Let \( \mu \) and \( \nu \) be two probability measures on a Polish space \( X \) and \( c : X \times X \to [0, \infty) \) a cost function induced by the distance on \( X \) i.e. \( c(x, y) = d(x, y) \). If \( \Pi \) is optimal for the Monge-Kantorovich problem between \( \mu \) and \( \nu \) with respect to the cost \( c \), then we can find a \( \mu \)-measurable \( 1 \)-Lipschitz map \( u : X \to X \) such that:

\[
\begin{cases}
  u(x) - u(y) = c(x, y) & \forall (x, y) \in \text{Supp}(\Pi) \\
  u(x) - u(y) \leq c(x, y) & \text{otherwise}
\end{cases}
\]

(8)

Because in our case the cost we are interested in is a distance \(|.|\) over \( H \), we consider a map \( u \) taken from this Proposition 3.3. It is worth to notice that the Problem (??) is the same as the following

\[
\min_{\Pi \in C(\rho_0, \rho_1)} \int_{H \times H} \beta(x, y) d\Pi(x, y),
\]

where the cost \( \beta \) is defined by

\[
\beta(x, y) := \begin{cases}
  \alpha(x - y) & \text{if } u(x) - u(y) = \|x - y\|_\infty \\
  +\infty & \text{otherwise}
\end{cases}
\]

(9)

We complete this section with the following Lemma, which is proved in [?] and easily adaptable in our setting.

Let \( \rho_0 \) and \( \rho_1 \) be two Borel probability measures on \( W \).

**Lemma 3.4** If \( \Pi \in O_2(\rho_0, \rho_1) \) then \( \Pi \) is concentrated on some \( \sigma \)-compact set \( \Gamma \) satisfying:

\[
\forall (x, y), (x', y') \in \Gamma, \quad x \in [x', y'] \Rightarrow (\nabla \alpha(y - x') - \nabla \alpha(y' - x), x - x') \geq 0.
\]

(10)

**Proof.** Since \( \Pi \) is a solution of (2), there is a Borel subset \( \Gamma \) of \( H \times H \) which is \(|.|\)-cyclically monotone. By inner regularity, up to remove a Borel set of zero measure, we can take \( \Gamma \) \( \sigma \)-compact. According to Proposition 3.3, we can find a potential \( u : H \to H \) such that:

\[
\forall (x, y) \in \Gamma, \quad u(x) - u(y) = |x - y|.
\]

Let \((x, y), (x', y') \in \Gamma \) such that \( x \in [x', y'] \). We have then:

\[
\begin{align*}
  u(x) &= u(y) + |x - y|, \\
  u(x') &= u(y') + |x' - y'|,
\end{align*}
\]

and since \( x \in [x', y'] \), we also have:

\[
|x' - y'| = |x - x'| + |x - y'|.
\]

Our potential \( u \) is a \( 1 \)-Lipschitz map, so:

\[
u(x') = u(y') + |x - x'| + |x - y'| \geq u(x) + |x - x'| \geq u(x').
\]
This equality leads to:
\[
 u(x') = u(x) + |x - x'| = u(y) + |x - y| + |x - x'| \\
 \geq u(y) + |y - x'| \geq u(x').
\]

With the previous notation, it turns out that \( \beta(x', y) = \alpha(x' - y) \) and \( \beta(x, y') = \alpha(x - y') \). Moreover thanks to Proposition 3.2, we also know that \( \Pi \) is \( \beta \)-cyclically monotone hence by symmetry of \( \alpha \):
\[
 \alpha(y - x) + \alpha(y' - x') \leq \alpha(y' - x) + \alpha(y - x').
\]

But by convexity of \( \alpha \), we have:
\[
 \alpha(y - x) - \alpha(y - x') \geq \nabla \alpha(y - x').(x' - x), \\
 \alpha(y' - x) - \alpha(y' - x') \leq -\nabla \alpha(y' - x).(x - x').
\]

So combining these inequalities with the \( \alpha \)-monotonicity we get:
\[
 (\nabla \alpha(y - x') - \nabla \alpha(y' - x), x - x') \geq 0.
\]

\[\square\]

**Remark 3.5** As in \[8\] the only reason to deal with \( \sigma \)-compact set \( \Gamma \), is that the projection \( P_1(\Gamma) \) is also \( \sigma \)-compact, and in particular a Borel set.

### 4 Structure of the support of some element of \( \mathcal{O}_2(\rho_0, \rho_1) \)

Throughout this part, *Differentiation theorem* 1.2 is used many times. We will present results in general framework. We consider \( \Pi \in C(\rho_0, \rho_1) \) and \( \Gamma \subset W \times W \) a \( \sigma \)-compact set on which \( \Pi \) is concentrated. For all the sequel we assume that \( \rho_0 = f \mu \) (the first measure has a density \( f \) w.r.t. \( \mu \)).

Let us fix a sequence of positive number \((\delta_p)_p\) which tends to 0 when \( p \) goes to infinity.

The following Lemma is a reinforcement of the one in \[8\] (Lemma 3.3).

**Lemma 4.1** Let \((y_n)_n\) be a dense sequence in \( H \). Then we can find a Borel subset \( D(\Gamma) \) on which \( \Pi \) is still concentrated and such that for all \((x, y) \in D(\Gamma), \forall r > 0, \) there exist \( n, k \in \mathbb{N} \) satisfying \( y \in B(y_n, \frac{1}{k+1}) \subset B(y, r), x \in \text{Leb}(f) \cap \text{Leb}(f_{n,k}) \) and for all \( p \in \mathbb{N} \):
\[
 \|f_{n,k}|_{B(x,\delta_p)}\|_{L^\infty} > 0,
\]
where \( f_{n,k} \) is the density of \((P_1)_\pi \Pi|_{H \times B(y_n, \frac{1}{k+1})}\).

**Proof.** Let \( \delta = \delta_p > 0 \) be fixed. We can find a recovering of \( H \) with countably balls \( \{B(x_m^{(p)}, \delta/2)\}_m \). For any \((n, k) \in \mathbb{N}^2 \) we consider \( f_{n,k} \) the density of the first marginal of the restriction of \( \Pi \) to \( H \times B(y_n, \frac{1}{k+1}) \) w.r.t. \( \mu \). Fix \( n, k \in \mathbb{N} \) and consider
\[
 D_{n,k}(\delta) := \left( \bigcup_{m \in \mathbb{N}} \{ x \in B(x_m^{(p)}, \delta/2), \|f_{n,k}|_{B(x,\delta)}\|_{L^\infty} = 0 \} \right) \times B(y_n, \frac{1}{k+1}).
\]

It turns out that
\[
 \Pi(D_{n,k}(\delta)) \leq \sum_{m \in \mathbb{N}} \int_{B(x_m^{(p)}, \delta/2) \setminus \{\|f_{n,k}|_{B(x,\delta)}\|_{L^\infty} > 0\}} f_{n,k}(x) d\mu(x) = 0.
\]
Besides since $\rho_0 \ll \gamma$ it is straightforward to see that for $C_{n,k} := H \setminus (\text{Leb}(f) \cap \text{Leb}(f_{n,k})) \times H$, 
\[ \Pi(C_{n,k}) = \rho_0 (H \setminus (\text{Leb}(f) \cap \text{Leb}(f_{n,k}))) = 0. \]

Therefore $\Pi$ is concentrated on the set $D_\delta(\Gamma) := \Gamma \setminus (\cup_{n,k}(D_{n,k}(\delta) \cup C_{n,k}))$.

Since $(\delta_p)_p$ is a countably sequence, it follows $D(\Gamma) := \cap_p D_{\delta_p}(\Gamma)$ has the desired properties. Indeed for any $\delta > 0$ if $(x,y) \in D_{\delta_p}(\Gamma)$, by density we can find $m,n,k \in \mathbb{N}$ such that $x \in B(x^{(p)}_m, \delta_p/2), y \in B(y_n, 1/(k+1)) \subset B(y, r)$. The result ensues.

Notice that the previous result is quite general, because it is true for any coupling, not necessarily optimal.

**Definition 4.2** Let $\Gamma$ be a $\sigma$-compact subset of $H \times H$. For $y \in \Omega$ and $r > 0$ we define:
\[ \Gamma^{-1}(\bar{B}(y, r)) := P_t(\Gamma \cap (H \times \bar{B}(y,r))). \]

An element $(x,y)$ of $\Gamma$ is called $\Gamma$-regular point if $x$ is a Lebesgue point of $\Gamma^{-1}(\bar{B}(y, r))$ for any $r > 0$.

It is worth noting that from the definition (4.2), for all measurable subset $A$ of $W$:
\[ \Pi(A \times \bar{B}(y, r)) = \Pi(A \cap \Gamma^{-1}(\bar{B}(y, r)) \times \bar{B}(y, r)). \]

**Lemma 4.3** Under assumptions of Lemma 4.1, any element of $D(\Gamma)$ is a $\Gamma$-regular point, namely:
\[ (x,y) \in D(\Gamma) \implies \lim_{\delta \to 0} \frac{\mu(\Gamma^{-1}(\bar{B}(y, r)) \cap B(x, \delta))}{\gamma(B(x, \delta))} = 1. \]

For the sequel, we introduce the following notation: if $\Gamma \subset H \times H$ then $T(\Gamma) = \{(1-t)x + ty, \ (x,y) \in \Gamma \}$. Since $\Gamma$ is $\sigma$-compact, $T(\Gamma)$ is $\sigma$-compact as well.

**Proposition 4.4** Let $\rho_0, \rho_1 \in D(\text{Ent}_\mu)$, and $\Pi \in \overline{\sigma}_2(\rho_0, \rho_1)$ concentrated on a $\sigma$-compact set $\Gamma$. Then for all $(x,y_0), (x,y_1)$ belonging to the set $D(\Gamma)$ obtained in the Lemma 4.1, with $y_0 \neq y_1$ and $\forall r > 0$ taken such that the closed balls centered at $y_0$ and $y_1$ with radius $r$ are disjoint, it holds:
\[ \gamma (T(\Gamma (\cap (B(x, \delta_p) \times B(y_0, r)))) \cap \Gamma^{-1}(\bar{B}(y_1, r)) \cap B(x, 2\delta_p)) > 0, \]
$\forall p \in \mathbb{N}$ large enough.

**Proof.** Let $f$ be the density of $\rho_0$ w.r.t. $\mu$. Consider $\Pi \in \overline{\sigma}_2(\rho_0, \rho_1)$ and let $(x,y_0), (x,y_1) \in D(\Gamma)$ such that $y_0 \neq y_1$. We can assume that $x \neq y_0$. We fix $r > 0$ for that $\bar{B}(y_0, r) \cap \bar{B}(y_1, r) = \emptyset$. Thanks to the discussion above (Lemma 4.1), we introduce $n_0, n_1, k \in \mathbb{N}$ such that $B(y_{n_0}, \frac{1}{k+1}) \subset B(y_0, r), B(y_{n_1}, \frac{1}{k+1}) \subset B(y_1, r)$. Since $\delta_p$ decreases to 0, we find $p \in \mathbb{N}$ large enough so that $0 < \delta = \delta_p < |x - y_0| + r$, and
\[ \gamma (B(x, \delta) \cap \Gamma^{-1}(\bar{B}(y_0, r)) \cap \Gamma^{-1}(\bar{B}(y_1, r))) > 0. \] (11)

This latter fact is possible thanks to the Proposition 4.3. The corresponding densities given by Lemma 4.2 are denoted by $f_{n_0,k}, \ f_{n_1,k}$.

Let us consider the Borel (up to a negligible set) set
\[ G_x := \{ z \in B(x, \delta), \ f_{n_0,k}(z) > 0, \ f_{n_1,k}(z) > 0 \}. \]
It turns out that $\mu(G_x) > 0$. Indeed according to Lemma 7.

$$\|f_{n_0,k}\|_{B(x,\delta)} \|_{L^\infty} > 0,$$

$$\|f_{n_1,k}\|_{B(x,\delta)} \|_{L^\infty} > 0.$$ 

Moreover we have 

$$\int_{\Gamma^{-1}(B(y_0,r))\cap B(x,\delta)} f_{n_0,k}d\mu > 0,$$

$$\int_{\Gamma^{-1}(B(y_0,r))\cap B(x,\delta)} f_{n_1,k}d\mu > 0.$$ 

The claim ensues thanks to (11).

Because $f_{n_1,k}$ is the density of $(P_1)_\# \Pi_{|W \times \tilde{B}(y_0,1/r)}$ we notice that:

$$\Pi \left( G_x \times \tilde{B}(y_1, \frac{1}{k+1}) \right) = \Pi \left( G_x \cap \Gamma^{-1}(\tilde{B}(y_1, \frac{1}{k+1})) \times \tilde{B}(y_1, \frac{1}{k+1}) \right)$$

hence 

$$\int_{G_x} f_{n_1,k}d\mu = \int_{G_x \cap \Gamma^{-1}(\tilde{B}(y_1,1/r))} f_{n_1,k}d\mu > 0.$$ 

It follows that 

$$\gamma(G_x \cap \Gamma^{-1}(\tilde{B}(y_1, r))) \geq \gamma \left( G_x \cap \Gamma^{-1}(\tilde{B}(y_1, \frac{1}{k+1})) \right) > 0. \quad (12)$$

Let $A(\delta):= B(x,2\delta) \cap \Gamma^{-1}(\tilde{B}(y_1, r)) \cap \Gamma \cap (B(x,\delta) \times B(y_0, r)).$

Consider the set $A_x := G_x \times B(y_{n_0}, \frac{1}{k+1}),$ and denote by $\Pi_{A_x}$ the restriction of $\Pi$ on $A_x$. We fix now $t \in (0, \frac{\delta}{\|x-y_0\|_{\infty}+r})$ so that: if $z \in B(x,\delta)$ and $w \in B(y_0, r)$ then $(1-t)z + tw \in B(x,2\delta)$. Indeed

$$|(1-t)z + tw - x| \leq (1-t)|z - x| + t|w - x| \leq |z - x| + t(|w - y_0| + |y_0 - x|) \leq \delta + \delta = 2\delta.$$ 

Therefore if we define $\rho_{t}^{A_x} := ((1-t)P_1 + tP_2)_\# \Pi_{A_x},$ firstly we have:

$$(P_1)_\# \Pi_{A_x}(G_x) \leq (P_1)_\# \Pi_{A_x}(B(x,\delta)) \leq \rho_{t}^{A_x}(B(x,2\delta))$$

and thus:

$$(P_1)_\# \Pi_{A_x}(G_x \cap \Gamma^{-1}(\tilde{B}(y_1, r))) \leq \rho_{t}^{A_x}(B(x,2\delta) \cap \Gamma^{-1}(\tilde{B}(y_1, r))).$$

Secondly thanks to (12):

$$(P_1)_\# \Pi_{A_x}(G_x \cap \Gamma^{-1}(\tilde{B}(y_1, r))) = \Pi \left( G_x \cap \Gamma^{-1}(\tilde{B}(y_1, r)) \times \tilde{B}(y_{n_0}, \frac{1}{k+1}) \right) = \int_{G_x \cap \Gamma^{-1}(\tilde{B}(y_1,r/2))} f_{n_0,k}d\gamma > 0.$$ 

And we deduce

$$\rho_{t}^{A_x}(B(x,2\delta) \cap \Gamma^{-1}(\tilde{B}(y_1, r))) > 0. \quad (13)$$

On the other hand, notice that $\rho_{t}^{A_x}$ is concentrated on $T(\Gamma \cap (B(x,\delta) \times B(y_0,r))$ hence:

$$\rho_{t}^{A_x}(B(x,2\delta) \cap \Gamma^{-1}(\tilde{B}(y_1, r))) = \rho_{t}^{A_x}(B(x,2\delta) \cap T(\Gamma \cap (B(x,\delta) \times B(y_0, r))) \cap \Gamma^{-1}(\tilde{B}(y_1, r))).$$
Combining this latter fact with (13), we get:
\[ \rho_1^A(A(\delta)) > 0. \]
And we know that \( \rho_1^A \) inherits of the convexity property, so is absolutely continuous w.r.t. \( \gamma \). Hence it implies \( \gamma(A(\delta)) > 0. \)

\[ \square \]

It is worth to notice that the Lebesgue differentiation theorem (Theorem 1.2) is only used to get the positivity in (11). This is provided by the Proposition 4.3, which needs this theorem. Without the theorem 1.2, the set considered in (11) is still non empty because it contains \( x \), but it can be of null measure.

5 Proof of the main theorem and comments

This section is devoted to prove Theorem 1.1.

**Theorem 5.1** Let \( \rho_0, \rho_1 \in D(\text{Ent}_p) \). If (1.1) is finite for some coupling, then any element of \( \overline{O}_2(\rho_0, \rho_1) \) is induced by a map \( T \), and therefore \( \overline{O}_2(\rho_0, \rho_1) \) is reduced to one element.

**Proof.** Let \( \Pi \in \overline{O}_2(\rho_0, \rho_1) \). In particular \( \Pi \in \mathcal{O}_2(\rho_0, \rho_1) \) and is concentrated on a \( \sigma \)-compact set \( \Gamma \) satisfying (10). Furthermore Lemma 4.1 provides us a \( \sigma \)-compact set \( D(\Gamma) \) on which \( \Pi \) is still concentrated. We claim that \( D(\Gamma) \) is contained in a graph of some Borel map. Let \((x_0,y_0)\) and \((x_0,y_1)\) in \( D(\Gamma) \) and suppose that \( y_0 \neq y_1 \). We can also assume \( x_0 \neq y_0 \). By strict convexity of \( \alpha \) we have:
\[ ((y_1 - x_0) - (y_0 - x_0), \nabla \alpha(y_1 - x_0) - \nabla \alpha(y_0 - x_0)) > 0. \]
Hence either \((y_1 - x_0, \nabla \alpha(y_1 - x_0) - \nabla \alpha(y_0 - x_0))\) or \((y_0 - x_0, \nabla \alpha(y_0 - x_0) - \nabla \alpha(y_1 - x_0))\) is positive. So without lost of generality we assume that:
\[ (\nabla \alpha(y_1 - x_0) - \nabla \alpha(y_0 - x_0), y_0 - x_0) < 0. \]
By continuity of \( \nabla \alpha \) we can find \( r > 0 \) small enough so that:
\[ \forall x, x' \in B(x_0, r), \forall y' \in B(y_0, r), \forall y \in B(y_1, r) : (\nabla \alpha(y - x') - \nabla \alpha(y' - x), y' - x) < 0. \]
\[ r > 0 \) can be chosen so that the balls \( \bar{B}(y_0, r) \) and \( \bar{B}(y_1, r) \) are disjoint.

Applying Proposition 4.4 to \((x_0, y_0), (x_0, y_1)\) we get:
\[ \mu(T(\Gamma \cap (B(x_0, \delta) \times B(y_0, r/2))) \cap \Gamma^{-1}(\bar{B}(y_1, r/2)) \cap B(x_0, 2\delta_p)) > 0, \]
\[ \forall p \in \mathbb{N} \) large enough. As a consequence we can find a \( \delta = \delta_p \in (0, r/2) \) small enough in such a way that there exist \((x', y') \in \Gamma \cap B(x_0, \delta) \times B(y_0, r/2) \) and \( x \in [x', y'] \cap B(x_0, 2\delta) \) and \( y \) such that:
\[ (x, y) \in \Gamma \cap ([x', y'] \cap B(x_0, 2\delta)) \times B(y_1, r). \]
Since \( x \in [x', y'] \), we have \( x - x' = \frac{x - x'}{y' - x} (y' - x) \). So by (10), we have:
\[ (\nabla \alpha(y - x') - \nabla \alpha(y' - x), x - x') = \frac{x - x'}{|y' - x|} (\nabla \alpha(y - x') - \nabla \alpha(y' - x), y' - x) \geq 0, \]
which contradicts (14). We obtain \( y_1 = y_0 \).

The unicity ensues from the convexity of \( \overline{O}_2(\rho_0, \rho_1) \), by the usual argument. \[ \square \]
Let us make some comments.

We have proved that $\Omega_2(\rho_0, \rho_1)$ is reduced to one element. However we do not know if $\Omega_2(\rho_0, \rho_1)$ has a unique element.

In [?], the authors do not require the absolute continuity of $\rho_t$ because the Lebesgue measure is doubling and invariant by translations. Thanks to that they can obtain good bounds for $\rho_t$ (see Proposition 2.2 in [?]).

The fact that $\rho_1$ is absolutely continuous with respect to $\gamma$ is important for the section 2, but we could hope it is possible to show the absolute continuity of interpolations $\rho_t$ ($t < 1$) without to pass by section 2. If it would be the case, the theorem 1.1 would be true for any probability measure $\rho_1$.

The strategy presented through the paper is general in the sense that the Hilbert norm $\|\cdot\|$ could be replaced by any finite-valued norm $\|\cdot\|$ on the Hilbert space $H$. 

12